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Superstrings, Knots, and Noncommutative Geometry in $\mathscr{E}^{(\infty)}$ Space

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Within a general theory, a probabilistic justification for a compactification which reduces an infinite-dimensional spacetime $\mathscr{E}^{(\infty)}(n = \infty)$ to a four-dimensional one $(D_T = n = 4)$ is proposed. The effective Hausdorff dimension of this space is given by $\langle \dim_H \mathscr{E}^{(\infty)} \rangle = d_H = 4 + \phi^3$, where $\phi^3 = 1/[4 + \phi^3]$ is a PV number and $\phi = (\sqrt{5} - 1)/2$ is the golden mean. The derivation makes use of various results from knot theory, four-manifolds, noncommutative geometry, quasiperiodic tiling, and Fredholm operators. In addition some relevant analogies between $\mathscr{E}^{(\infty)}$, statistical mechanics, and Jones polynomials are drawn. This allows a better insight into the nature of the proposed compactification, the associated $\mathscr{E}^{(\infty)}$ space, and the Pisot–Vijayvaraghavan number $1/\phi^3 = 4.236067977$ representing its dimension. This dimension is in turn shown to be capable of a natural interpretation in terms of the Jones knot invariant and the signature of four-manifolds. This brings the work near to the context of Witten and Donaldson topological quantum field theory.

1. INTRODUCTION

As succinctly noted by Duff [1] "Unfortunately, superstrings have as yet no answer to the question of why our universe appears to be four dimensional, let alone why it appears to have a signature (3, 1)."

In the main part of the present work we intend to give an answer to this question. In the course of doing that, we will be utilizing and also discovering various analogies and some nontrivial relations between our probabilistic approach to "compactification" and several other branches of current research in pure and applied mathematics, in particular knot theory [2] and noncommutative geometry [3].

Our main thesis is that the dimensionality of spacetime, as occasionally speculated in the past by some notable scientists, is a derivable property

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akin to temperature [20]. In fact we will be showing that the apparent dimensionality of our every-day spacetime $D_T = 4$ is derivable from more primitive assumptions and follow from the same law of statistical distribution used by M. Planck to derive his well-known formula for blackbody radiation. This formula, we may recall, was, at least historically speaking, the beginning of quantum physics. Naturally, to undertake such analysis we will have to introduce some radical deviation from our classical notion of spacetime. The most decisive point in that respect is the introduction of randomness and scale invariance to the very concept of spacetime geometry, which in turn shows that spacetime loses its smoothness when we sharpen the resolution of observation as is essential for the micro spacetime of quantum physics. This spacetime, which we refer to as Cantorian space time $\mathcal{E}^{(\infty)}$ for obvious reasons, has some remarkable properties [4–13]. First, it is an infinite-dimensional hierarchical and random geometrical manifold with infinite numbers of equivalent paths (connections) between any two points. Second, any so-called point in this space will always reveal a structure on a close examination, so that strictly speaking the concept "point" does not exist in $\mathcal{E}^{(\infty)}$, which is a resolution-dependent zoom space. It then turns out that $\mathcal{E}^{(\infty)}$ is basically a form of noncommutative geometry. The simplest and best-studied example for such a geometry would be the famous Penrose tiling. It is therefore not surprising that Penrose tiling was presented from the very beginning as a low dimensional example for $\mathcal{E}^{(\infty)}$ as well as NCG which obeys a noncommutative C*-algebra [3].

Another point of importance is the connection between the basic formulas defining $\mathcal{E}^{(\infty)}$ and the Jones knot polynomial V_L [8]. In fact it turned out that a fundamental upper bound for V_L is given by a formula which is a special form of the basic bijection formula of $\mathcal{E}^{(\infty)}$ and that the well-known Hausdorff dimension of a quantum path $d_H = 2$ may be derived from the properties of knots in R^4. Finally we will show that the set of Penrose tiling and quasi crystallography provides a unique space where the classical and quantum descriptions of spacetime meet [6]. In [21] some proposals for experimental verrification of $d_H = 2$ are given.

2. THE CANTORIAN SPACE $\mathcal{E}^{(\infty)}$

The basic conept and equations of the space $\mathcal{E}^{(\infty)}$ were introduced in some detail on numerous previous occasions, so that we may confine ourselves to a short introduction coupled with a summary of the main equations [4–13]. For simplicity we may start with Mauldin–Williams theorem, which states that with a probability one, a randomly constructed triadic Cantor set $\mathcal{S}_c^{(0)}$ will have a Hausdorff dimension equal to the golden mean

$$\dim_H S_c^{(0)} = d_c^{(0)} = \phi = (\sqrt{5} - 1)/2 \tag{1}$$

The question is now, how can we lift this formula to n dimensions, i.e. what is the Hausdorff dimension of $\mathcal{S}_c^{(n)}$. It turned out that a very simple formula is all that we need. This formula is a generalization of the relationship between the triadic Cantor set $d_c = \ln 2/\ln 3$ and the Sierpinski gasket $d_s = 1/d_c = 1/(\ln 2/\ln 3) = \ln 3/\ln 2$ and is termed the bijection formula [4–13]

$$\dim_H S_c^{(n)} = d_c^{(n)} = \left(1/\dim_H S_c^{(0)}\right)^{n-1} = (1/d_c^{(0)})^{n-1} \tag{2}$$

Consequently, in four dimensions, the Mauldin–Williams theorem is

$$\dim_H \mathcal{S}_c^{(4)} = (1/\phi)^{(4-1)} = \frac{1}{\phi^3} = 4 + \phi^3 = 4.236067977 \tag{3}$$

where $1/\phi^3$ is known as the Pisot–Vijayvaraghavan number [19].

Next let us construct the $\mathcal{E}^{(\infty)}$ space *ab initio*. We do that using an infinite number of Cantor sets $S_c^{(0)}$ with all conceivable Hausdorff dimensions in the unit interval. Suppose these sets are all "mixed" together in all possible forms of union and intersections to form one large space made of infinite weighted dimensions. Now we ask the following question. What is the expectation value for the dimensionality of this formally infinite dimensional space? To answer this question we need to know the distribution function according to which it was constructed. Assuming a gamma distribution Γ, the expectation value of this Gaussian distribution is [4–13]

$$E_\Gamma(n) = \langle n \rangle = \bar{r}/\lambda \tag{4}$$

Setting the shape factor $\bar{r} = 2$ and substituting for the mean value of a Poisson distribution of the elementary Cantor sets, i.e., $\lambda = \ln(1/d_c^{(0)})$ in $E_\Gamma(n)$, one finds

$$\left\langle \dim_T \mathcal{E}^{(\infty)} \right\rangle = \langle n \rangle = 2/\ln(1/d_c^{(0)}) = \dim \mathcal{H}/\ln\left(\dim_H S_c^{(2)}\right) \tag{5}$$

where $\mathcal{H}$ is the Hilbert space of Witten theory for two spheres with four points.

Expanding and retaining the linear terms only, we obtain

$$\sim\left\langle \dim_T \mathcal{E}^{(\infty)} \right\rangle = \sim\langle n \rangle = (1 + \dim_H S_c^{(0)})/(1 - \dim_H S_c^{(0)}) = \frac{1 + d_c^{(0)}}{1 - d_c^{(0)}} \tag{6}$$

It was shown in previous work that the last expression is exact within a genuinely discrete space [12].

Now, to have a space filling it is clear that we must satisfy the following condition [12]:

$$\sim\left\langle \dim_{T} \mathscr{E}^{(\infty)} \right\rangle = \sim\langle n\rangle = d_c^{(n)} = (1/d_c^{(0)})^{n-1} \tag{7}$$

It is an elementary matter to solve the above equation and show that it can be satisfied iff $d_c^{(0)} = \phi = (\sqrt{5} - 1)/2$ and $n = 4$.

In other words, the expectation value for the dimensionality of $\mathscr{E}^{(\infty)}$ is identical to Mauldin–Williams theorem in four dimensions. Consequently our $\mathscr{E}^{(\infty)}$ space, although of infinite dimension, has an effective finite expectation value for the topological dimension ($n = 4$) and appears therefore as if it were four dimensional. This is a very similar situation to that of a Bethe lattice [9]. It is not particularly difficult to show that the same space also has an expectation value for the Hausdorff dimension given by

$$\left\langle \dim_{H} \mathscr{E}^{(\infty)} \right\rangle = \langle d_c\rangle = \sim\left\langle \dim_{T} \mathscr{E}^{(\infty)} \right\rangle = \frac{1}{(1 - d_c^{(0)})\, d_c^{(0)}} = \frac{1}{\phi^3} \tag{8}$$

The above equation will be shown in Section 4 to correspond to equation (21) of the theory of subfactors [3].

We have also given elsewhere some arguments for the fact that the signature of spacetime must be perceived as (3, 1), which we will not repeat here [12].

To sum up, Cantorian spacetime $\mathscr{E}^{(\infty)}$ is infinite dimensional but has an effective Hausdorff dimension $d_H = \langle d_c\rangle \simeq 4$, while the topological dimension of the core is exactly 4. We have thus reduced the infinite dimension to a hierachical four-dimensional space. The idea is very different from the original three main methods of compactification used in superstrings, namely toroidal compactification, orbifold compactification, and Calabi–Yau compactification. There, a finite internal manifold is left over after compactification where the vibration of the strings take place. Such a manifold could be a six-dimensional orbifold, for instance. A detailed discussion of our approach to this point will be given elsewhere.

It seems therefore that we in the macroworld are aware of the four dimensions only, while the infinite rest are very unlikely to be observed in a sense similar to the extremely tiny compactified dimensions of all types of Kaluza–Klein theories. It is important to note the remarkable continuous fractal representation of $\sim\langle n\rangle$, namely

$$\sim\langle n\rangle = 4 + \phi^3 = 4 + (\bar{4}) = 1/(\bar{4}) \tag{9}$$

It suggests an imaginative picture of a 4D fractal universe containing a much smaller 4D fractal universe and so on ad infinitum. We may mention that the preceding analysis was inspired by an old proposal due to A. Wheeler. For more details of the analysis and a discussion of its implications we refer the reader to refs. 4–13.

3. KNOT THEORY AND $\mathscr{E}^{(\infty)}$

A knot is defined as a smooth-embedding of a circle in R^3 [2]. An important result in this subject is the discovery of the following index by Jones [3]:

$$J_{ind} = (2 \cos \pi/r)^2 \tag{10}$$

It is easily verified that for $r = 5$ we find J_{ind} to be numerically identical to the Hausdorff dimension of a three-dimensional Cantorian space $s_c^{(3)}$ when the null set, i.e., the kernal $d_c^{(0)}$ of $\mathscr{E}^{(\infty)}$, is taken to be $d_c^{(0)} = \phi$. This is because

$$J_{ind}\Big|_{r=5} = (2 \cos \pi/5)^2 = 2 + \phi \tag{11}$$

while the bijection formula [4–13]

$$\dim_H S_c^{(n)} = d_c^{(n)} = (1/d_c^{(9)})^{n-1} \tag{12}$$

gives the same result for $n = 3$ and $d_c^{(0)} = \phi = (\sqrt{5} - 1)/2$.

Another very important result also found by Jones is the following [14]. If an oriented link L is a closed n-braid, then

$$\left|V_L e^{2\pi i/r}\right| \leq (2 \cos \pi/r)^{n-1}; \qquad r = 3, 4, \ldots \tag{13}$$

It is equally easily verified that the right-hand side of (13) is nothing else but our familiar bijection formula of Cantorian space $\mathscr{E}^{(\infty)}$ and that for $n = 4$ and $r = 5$ one finds [14, 8]

$$\left|V_L e^{2\pi i/5}\right| \leq (2 \cos \pi/5)^3 = (1/d_c^{(0)})^3 = \left\langle \dim_H \mathscr{E}^{(\infty)} \right\rangle = 4 + \phi^3 \tag{14}$$

when setting $d_c^{(0)} = \phi$. Here we continue to think of n as the topological dimension of $\mathscr{E}^{(\infty)}$, while $r = K + 2$ is a parameter, where K can be thought of as the inverse of the Planck constant $\hbar$. Equation (13) thus gives the expectation value of the dimension of the infinite-dimensional hierachical Cantorian space in this particular case, which is [4–13]

$$\sim\langle n\rangle = \langle d_c\rangle = 4 + (\bar{4}) = 4 + \phi^3 = 2 + \sqrt{5} \tag{15}$$

Note that in case of $r = 5$ and $n = 3$ we have $\left|V_L e^{2\pi i/5}\right|. = J_{ind}$ where $\big|.$

means evaluated at the corresponding parameters. Note also that for $n = 4$ we must have $1 \leq \left|V_L e^{2\pi i/r}\right| \leq 8$ with $1/2 \leq d_c^{(o)} \leq 1$.

4. NONCOMMUTATIVE GEOMETRY (NCG) AND CANTORIAN SPACETIME

Noncommutative algebra is a basic tool in mathematical physics, with its most important application in quantum mechanics. The aim of noncommutative geometry is to extend the idea to algebraic geometry. The spacetime concept of NCG and Cantorian spacetime (CST) have many common features [8].

For instance, one of the topological invariants of a certain very interesting noncommutative or "quantum" space X, which is the dimension group, is a subgroup of $\mathbb{R}$ generated by $\mathbb{Z}$ and the inverse of the golden mean $1/\phi = 1 + \phi = \sqrt{5} + 1/2$. It is followed then that a certain dimension [3]

$$\dim(e) = \tau(e) \tag{16}$$

can take only values in the subgroup [3]

$$\mathbb{Z} + \left(\frac{1}{\phi}\right)\mathbb{Z} \tag{17}$$

Following ref. 3, it can be shown that there is a semigroup K_0 such that

$$K_o^+(A) = \left\{(n, m) \in Z^2, \qquad n\left(\frac{1}{\phi}\right) + m \geq 0\right\} \tag{18}$$

It is clear from the last equation that for $n = 2$ and $m = 1$ we have

$$n\left(\frac{1}{\phi}\right) + m = 2\left(\frac{1}{\phi}\right) + 1 = 4 + \phi^3 \tag{19}$$

which is identical to what we have obtained for $d_c^{(4)} = \langle \dim_H \mathscr{E}^{(\infty)} \rangle$ in Equations (3) and (9). Furhtermore and following the theory of subfactors and the notation of ref. 3, it can be shown that the index $[M: N]$ of N in M [3]

$$[M: N] = \dim_N (L^2(M)) \tag{20}$$

is also given by

$$\dim_N (L^2(M)) = \frac{1}{(1 - \lambda_0)\lambda_0} \tag{21}$$

where $\lambda_0 = \mathrm{Tr}\, M(e)$. It is clear from the right-hand side of (21) that this expression is nothing but the formula for the expectation value of the Hausdorff dimension of $\mathscr{E}^{(\infty)}$, namely [4–13]

$$\left\langle \dim_{H} \mathscr{E}^{(\infty)} \right\rangle = \langle d_c \rangle = 1/[(1 - d_c^{(0)})\, d_c^{(0)}] \tag{22}$$

and we just need to set

$$\lambda_0 = \dim_{H} S_c^{(0)} = d_c^{(0)} = \phi = (\sqrt{5} - 1)/2 \tag{23}$$

in order to find again our by-now-familiar number

$$[M: N] = \langle d_c \rangle = \frac{1}{(1 - \phi)\phi} = \frac{1}{\phi^3} = 4 + \phi^3 \tag{24}$$

It should be noted that minimizing $\langle d_c \rangle$ leads to $d_c^{(0)} = 1/2$ and $\langle d_c \rangle_{\min} = 4 = \langle \dim_H \mathscr{E}^{(\infty)} \rangle_{\min}$.

Another interesting correspondence between the formalism of $\mathscr{E}^{(\infty)}$ and that of NCG is evident from looking at the expression for t as given on 59 in ref. 3 for Potts modle. There we see that setting $q = \phi$ or $1/\phi$ in

$$\bar{t} = (2 + q + q^{-1})^{-1} \tag{25}$$

leads to

$$\bar{t} = \phi^3 \tag{26}$$

and consequently

$$1/\bar{t} = [M:N] = \langle d_c \rangle = 2 + \sqrt{5} \tag{27}$$

A similar result is also obtained for the Jones polynomial of the unlink for $t = \phi$. This means

$$V_u = (\sqrt{t} + 1/\sqrt{t})\Big|_{\phi}^{2} = 4 + (\bar{4}) = [M:N] = 1/\bar{t} = 2 + \sqrt{5}$$

5. CONNECTIONS TO FOUR-MANIFOLDS

To illustrate the connections between the geometry of four-manifolds and $\mathscr{E}^{(\infty)}$ consider first the signature τ [15]

$$\tau = b^+ - b^- \tag{28}$$

where $b^+ = \dim H^+$ and $b^- = \dim H^-$ are the dimensions of the maximal positive and negative suspaces of the form H_2, respectively [15]. Setting

$$b^+ = \dim_{H} \operatorname{Ker} \mathscr{E}^{(\infty)} = \phi \tag{29}$$

and

$$b^{-} = \dim_{H} \text{CoKer } \mathcal{E}^{(\infty)} = 1 - \phi = \phi^2 \tag{30}$$

one finds from (28) that [8]

$$\tau = [b^{+} \ominus b^{-}]_{\phi} = [b^{+} \otimes b^{-}]_{\phi} = \phi - \phi^2 = \phi\phi^2 = \phi^3 \tag{31}$$

Recalling that

$$\tau = [M{:}N]^{-1} \tag{32}$$

then

$$\dim_{N} (L^2(M)) = [M{:}N] = 1/\tau = \left\langle \dim_{H} \mathcal{E}^{(\infty)} \right\rangle = 4 + \phi^3 \tag{33}$$

This is exactly the same result obtained earlier [4–9].

It is worth mentioning here that in ref. 16 it was found that the Fibonacci numbers play a role in the topology of four-manifolds. The importance of the PV number $2 + \sqrt{5} = 4.23606\ldots$ seems to extend far beyond quasicrystallography, where it found one of its first physical application [8]. The most important conclusion so far, is however, that the Jones knot invariant has a natural interpretation in terms of the dimensionality of $\mathcal{E}^{(\infty)}$ and the signaute of four-manifolds.

6. QUASICRYSTALLOGRAPHY AND $\mathcal{E}^{(\infty)}$

One of the surprises which we have encountered recently is that the expectation value of the effective dimensionality of $\mathcal{E}^{(\infty)}$, namely $\sim\langle n\rangle = 4 + \phi^3 = 2 + \sqrt{5}$, as well as its inverse ϕ^3 crop up in connection with the theory of quasicrystallography. In ref. 17 it was found that the Z-module which carries the diffraction pattern possesses a certain symmetry which is invariant through a group of homotheties. Noting that Shur's lemma entails under certain conditions that

$$\Lambda\pi^{\parallel} + \Lambda'\pi^{\perp} \tag{34}$$

for any real numbers Λ and Λ', one finds that

$$M = \phi^{-3}\pi^{\parallel} - \phi^3\pi^{\perp} \tag{35}$$

where $\pi^{\|}$ and $\pi^{\perp}$ are certain projective matrieses [17] and

$$\phi^{-3} = 4 + \phi^{3} = 2 + \sqrt{5} = \langle d_c \rangle = [M:N] \tag{36}$$

7. PENROSE UNIVERSES, NCG, AND $\mathscr{E}^{(\infty)}$

One of the most important results in noncommutative geometry is undoubtedly the conclusion by Connes that the Penrose space X represents in effect an example of a low-dimensional noncommutative space [3].

To appreciate the importance of this result in nonclassical physics we need just recall that while classical mechanics obeys commutative algebra, quantum mechanics in the Heisenberg–Born formalism is manifestly a problem in noncommutative analysis. Thus, the Penrose universe is a unique medium where classical and nonclassical physics meets. The objective of this section is to show the intimate connection between Penrose universe and cantorian spacetime $\mathscr{E}^{(\infty)}$, an undertaking which may lead to a resolution of many paradoxes in quantum physics, by reducing them to the nonclassical transfinite nature of the ambient micro spacetime of quantum and subquantum particles. To see how important such an undertaking is as well as the importance of noncommutative geometry, it may be sufficient to mention that it is very likely that Einstein's opposition to quantum mechanics stemmed from the fact that all forms of geometries known to him at that time were commutative. The noncommutative quantum theory of Heisenberg and Bohr may have therefore remained obscure to him because it did not fit into his basically geometric thinking. In other words, it may be reasonable to suppose that had Einstein known about the possibility of noncommutative geometry, he would most probably have modified his attitude toward quantum mechanics [3].

With the experimental discovery of quasicrystals which possesses the supposedly forbidden fivefold symmetry, the subject of nonperiodic tiling of space acquired a prominent place in mathematical physics, particularly in the context of the work of R. Penrose and H. Conway on Penrose tiling [3].

The recent work of Connes [3] on noncommutative geometry added a new dimension to the importance of nonperiodic tiling by realizing that X is an example of noncommutative geometry. The starting point of this realization is to look at the case of measured foliations with continuous dimension [3]. The general von Neumann projection of the foliations gives a sort of a random Hilbert space. This is of course reminiscent of Mauldin's theorem and our golden mean theorem [4–13]. The space of Penrose tiling of the plane does indeed give a very clear geometrical intuitive picture of what a leaf of foliation could generally look like. The reason for that is simply the following [3]. When analyzing X using classical tools, it appears to be pathological, as observed by Connes. However when we replace the commutative C*-algebra with a noncommutative C*-algebra we find that X is readily analyzed [3]. This is a clear indication of the inherently quantum mechanical nature of the space X. In this sense we understand this space as an example of noncommutative geometry of a low-dimensional noncommutative space [3].

The preceding reasoning yields two quantitative results which imply quite unexpected connections of the Penrose space to knot theory and Cantorian spacetime. The first result is that X has a natural subfactor which is identical to the Jones index [3],

$$J_{ind} = (2 \cos \pi/5)^2 \tag{37}$$

This was discussed earlier as an important quantity in knot theory [2]. The second point is that J_{ind} is itself numerically identical to the Hausdorff dimension of a three-dimensional Cantorian space which represents a generalization of Mauldin's theorem [2] to three dimensions using the so-called bijection formula

$$d_c^{(3)} = (1/d_c^{(0)})^{(3-1)} = \left(\frac{1}{\phi}\right)^2 = 2 + \phi = J_{ind} \tag{38}$$

These are by no means the only indications that the Penrose universe can be seen as a realization of a projection of a four-dimensional Cantorian space with

$$\dim_H \text{Ker } \mathscr{E}^{(\infty)} = \dim (\varnothing \text{ of } \mathscr{E}^{(\infty)}) = \phi \tag{39}$$

To explain this point we suppose we have a circular region in X of diameter ρ. Suppose further that one is transferred to a randomly chosen parallel Penrose universe. Then we ask the following question: How far do we need to travel from our initial circular region in order to end in another circular region which can match the initial one? The answer is that we have to travel a distance

$$l_\rho \leq \left(\frac{4 + \phi^3}{2}\right)\rho = (2.118033989 \ldots)\rho \tag{40}$$

This is the essence of the local isomorphism theorem. This theorem is of course trivial for a periodic pattern, but the Penrose universe is nonperiodic and here is the first surprise. The second surprise is that $(4 + \phi^3)/2 = 2.118033989$ is exactly equal to half of $1/\phi^3$, which is the exact expectation value for the dimensionality of the hierachical Cantorian universe $\mathscr{E}^{(\infty)}$. As mentioned earlier, this essentially infinite-dimensional universe has an effective Hausdorff dimension $\langle \dim_H \mathscr{E}^{(\infty)} \rangle = 4 + \phi^3 = 4 + (4)$ and a topological dimension for the effective "core" of exactly $D = n = 4$, as can again be seen immediately from the bijection formula introduced earlier,

$$\dim_H S_c^{(n)} = \left((1/\dim_H S_c^{(0)} \right)^{n-1} \tag{41}$$

when setting $n = 4$ and $\dim_H S_c^{(0)} = d_c^{(0)} = \phi$

$$\dim_H S_c^{(4)} = d_c^{(4)} \left(1/d_c^{(0)} \right)^3 = (1/\phi)^3 = \left\langle \dim_H \mathcal{E}^{(\infty)} \right\rangle = 4 + \phi^3 \tag{42}$$

in agreement with our earlier discussion.

It is worth remembering and stressing again that the distribution used to obtain these results is the same distribution used to derive the formula of blackbody radiation. In this sense dimensionality seems to share indeed some essential features with temperature as anticipated by Finkelstein [20].

8. KNOT THEORY AND THE HAUSDORFF DIMENSION OF A QUANTUM PATH

As mentioned earlier, a knot is by definition a smooth embedding of a circle in r^3 [2]. A circle c_1 is of course a dim $c_1 = 1$ geometrical object. It can be deformed to a true knot only in a space S_3 where dim $S_3 = 3$. consequently the codimension must be

$$\begin{aligned} \text{Codim } C_i &= \dim S_3 - \dim C_1 \\ &= 3 - 1 = 2 \end{aligned} \tag{43}$$

There is however, no "real" knots in R^4 for which dim $S_4 = 4$ because all knots are equivalently trival and dissolve in this higher dimensional space. The only valid way to generalize the concepts of knot theory to higher dimensions is to keep the Codim = 2 constant. Consequently the geoemtrical object corresponding to the circle must have a dim $C_2 = 2$. Therefore in R^4 we must have

$$\text{Codim } C_2 = 4 - 2 = 2 \tag{44}$$

In other words, to have a knot in R^4 we must have

$$\dim C_2 = \text{Codim } C_2 = 2 \tag{45}$$

The next point is based on the hypothesis that all particles and all interactions between particles in microspace are a manifestation of fractal-like knots in the "fabric" of Cantorian spacetime $\mathscr{E}^{(\infty)}$ at an appropriate resolution. Now since $\mathscr{E}^{(\infty)}$ is effectively four dimension by virtue of

$$\sim\left\langle \dim_{\mathrm{T}} \mathscr{E}^{(\infty)} \right\rangle = \frac{\dim \mathrm{C}_2}{\dim \mathrm{CoKer}\, \mathscr{E}^{(\infty)}} - 1 = 5.23606 - 1 = 4 + \phi^3$$
$$\simeq 2 \otimes 2 = 2 \oplus 2 = 4 \qquad (46)$$

then it follows that we could not have a knot, a particle, and consequently a particle path in $\mathscr{E}^{(\infty)}$ unless $\dim \mathrm{C}_2 = 2$, and that means

$$\langle \mathrm{d}_c^{(2)} \rangle = \dim \mathrm{C}_2 = \mathrm{Codim}\ \mathrm{C}_2 = 2 \qquad (47)$$

In this sense a Cantorian spacetime sheet must fall back on itself and thus form an effective four-dimensional knot. That is basically why a quasicontinuous connection in $\mathscr{E}^{(\infty)}$ is essentially a path with a Hausdorff dimension $\langle \mathrm{d}_c^{(2)} \rangle = 2$ [4–13]. In this context two more side remarks are in order. First we recall the Frish–Wasserman–Delbrück conjecture, which states that the probability for a randomly embedded circle to be knotted tends to one as the length of the circle tends to infinity. For a "fractal" circle, the length of any part is infinite provided the circle appears to be continuous at the corresponding resolution of observation, and if the FWD conjecture is correct then a fractical circle is everywhere knotted. Furthermore, FWD implies also that a self-avoiding polygon must be knotted and since in 4 dimensions a polygon is naturally self-avoiding, this means our "circles" in $\mathscr{E}^{(\infty)}$ must be knotted. We will regard all forms and particle interactions as a manifestation of these transfinite-fractal knots as indicated earlier. Finally, we may recall that the univeres as a whole may be regarded in some speculative models as a knot complement. This concludes our topological justification of $\langle \mathrm{d}_c^{(2)} \rangle = 2$.

9. THE SO-CALLED WAVE–PARTICLE DUALITY, NCG, AND $\mathscr{E}^{(\infty)}$

The aim of the present section is to show that the indistinguishability theorem of Cantorian spaces $\mathscr{E}^{(\infty)}$ [7]

$$\overset{\oplus}{\Omega} = \overset{\oplus}{\Omega} = \phi^3, \qquad \phi = (\sqrt{5} - 1)/2 \qquad (48)$$

is conceptually homomorphic to a wave–particle interpretation of the index theorem of Toeplitz operators [13]

$$\mathrm{Ind}(\mathrm{T}(f)) = \langle [\eta] f^*[\mathrm{T}] \rangle \qquad (49)$$

In fact, it can be shown in an elementary fashion that our indistinguishability theorm (48) is derivable from the index theorem of (49), which we do next.

Let f be a complex continuous and never vanishing periodic function on $\mathbb{R}$ with a period 2π; then we can write

$$f(x) = e^{inx+\psi(x)} \tag{50}$$

where n can be interpreted as a winding number:

$$n = \frac{1}{2i\pi}\int_0^{2\pi} [(f'(x))/(fx)]\, dx \tag{51}$$

for the closed path

$$\gamma = f(T) \tag{52}$$

in the complex plane given by the image

$$T = \mathbb{R}/\mathbb{Z} \tag{53}$$

The winding number can be also expressed as

$$n = \int_\gamma \frac{1}{2i\pi Z}\, dZ = \int_\gamma \eta \tag{54}$$

where the closed 1-form η defines a de Rham cohomology class in the first cohomology group [13].

Similarly and since γ is a closed path, and therefore defines a homology class we have $[\gamma] = f^*[T]$. Consequently using de Rham duality, n can be written as

$$n \Rightarrow \overset{\otimes}{n} = -\langle[\eta]\, f^*[T] = -\langle[\eta][\gamma]\rangle \tag{55}$$

The right handside of the above equation (55) is thus analogus to the Born formula and reflects therefore the homological structure of the Schrödinger wave quantization. next let us state the well known result that the Toeplitz operator $T\,(f)$ is Fredholm [13]. Since a Fredholm operator admits a finite dimensional Kernel and co-Kernel we can state the well known formula

$$\mathrm{Ind}(T) = \dim(\mathrm{Ker}\,(T)) - \dim(\mathrm{CoKer}(T)) \tag{56}$$

This formula is clearly analogus to the index of four manifolds when we admit continuous dimensions [4–8]

Now we can state the index theorem

$$\mathrm{Ind}(T(f)) = -\langle[\eta][\gamma]\rangle = n \tag{57}$$

In words, this means, the index of a Toeplitz operator $T(f)$ is equal to the winding number n off. However, since the left hand side of (56) and (57) are a winding number in terms of an operator, it is analogus to the essentially particle picture of the Heisenberg-Born quantization formalism. Consequently we may write

$$n \Rightarrow \overset{\oplus}{n} = \mathrm{Ind}(T) \tag{58}$$

Next we like to give a derivation of the index theorem of equ. 49 using a purely formal analogy between the formalism of the index theorem and that of Cantorian spaces $\mathscr{E}^{(\infty)}$. This procedure which holds for knot theory, noncommutative geometry as well as four manifolds [10] will be referred to rather loosely as "analogical" continuation. Following ref. 15, equation (56) can be rewritten by performing the following replacements:

$$\mathrm{Ind}(T(f)) \to \tau = b^{+} - b^{-} \tag{59}$$

$$\dim(\mathrm{Ker}\ (T(f))) \to \dim(\mathrm{Ker}\ (\mathscr{E}^{(\infty)})) = b^{+} = \phi \tag{60}$$

$$\dim(\mathrm{CoKer}\ (T(f))) \to \dim(\mathrm{CoKer}(\mathscr{E}^{(\infty)})) = b^{-} = \phi^{2} \tag{61}$$

where $\phi = (\sqrt{5} - 1)/2$ is the golden mean,

Consequently one finds

$$\overset{\oplus}{n} = \mathrm{Ind}\ (T(F)) = \tau = \phi^{3} \tag{62}$$

ϕ^3 is the inverse of the well-nown expectation value for the dimension of $\mathscr{E}^{(\infty)}$, namely $\langle \dim_H \mathscr{E}^{(\infty)} \rangle = 4 + \phi^3$, which is a PV number.

On the other hand, making the following exchanges in equation (55)

$$[\eta] \to \langle -\phi^{2}| \tag{63}$$

and

$$[\gamma] \to \langle \phi| \tag{64}$$

one finds

$$\overset{\otimes}{n} = -\langle -\phi^{2}|\phi\rangle = \phi^{3} \tag{65}$$

Next we invoke the geometric probability interpretation of the Hausdorff dimension of the kernel and cokernel of $\mathscr{E}^{(\infty)}$ as discussed, for instance, in ref. 4. We see immediately that n as given by (56) and (62) is essentially the addition theorem of independent probability events, which can be consistent only with a particle picture.

On the other hand, equations (55) and (65) are a clear statement of the multiplication theorem, which make sense only for extended objects such as a Schrödinger wave and never for a particle. In other words

$$\overset{\oplus}{n} = \overset{\otimes}{n} \tag{66}$$

is entirely consistent with

$$\overset{\oplus}{\Omega} = \overset{\otimes}{\Omega} \tag{67}$$

as well as the wave–particle duality interpretation of the index theorem [5,18].

10. CONCLUSION

Starting with a transfinite hierachical spacetime $\mathscr{E}^{(\infty)}$ of infinite dimensions and following some ideas due to A. Wheeler, D. Finkelstein, and C. von Weizsäcker, we derive a finite expectation value and an effective Hausdorff dimension for $\mathscr{E}^{(\infty)}$ using a gamma distribution. This is the same distribution used to derive the Maxwell velocity distribution law as well as the Planck blackbody radiation formula. This may be seen as an indication for a conceptual link between temperature and dimensions. The transformation $\mathscr{E}^{(\infty)} \leftrightarrows \mathscr{E}^{(4)}$ can then be used as a basis for developing a form of transfinite hierachical superstring theory which is closely related to knot theory, noncommutative geometry, and quasiperiodic tiling. The hierachical dimensional formula $\langle d_c \rangle = 4 + (4) = 4.23\ldots$ is obtained for $\mathscr{E}^{(\infty)}$ without the need of supressing any terms such as the Lovelace term $[1 - (D - 2)/24]$ or invoking supersymmetry and it allows for a wide spectrum of a possible quantized "vibration" on finer and finer "dimensional" scales.

In addition, the relativistic demand on a minimum string's world surface can be met in our model in an elementary fashion by minimizing $\langle d_c \rangle$ as given by equations (21) and (22). The result follows then from Differ $\langle d_c \rangle = 0$ to $d_c^{(0)} = 1/2$ and consequently $\langle d_c \rangle_{min} = 4$ in a natural way.

Important insight into the meaning of the connection between topological quantum field theory, knots, and $\mathscr{E}^{(\infty)}$ may be gained by contemplating the meaning of n in both equation (2) for dim $S_c^{(n)}$ and equation (13) for $|V_L e^{2\pi i/r}|$. In the first, n clearly denotes topological dimension. In the second, the meaning of n is slightly more involved. An n-braid is a braid group on n strands B_n. This B_n can be defined formally as a fundamental group in configuration space C_n of n distinct points. The braid can then be viewed as the spacetime graph of motion along a closed connection in C_n and that establishes the analogy to our $\mathscr{E}^{(\infty)}$ spacetime. The next step is to look at the classical limit of the corresponding TQF theory by letting $\hbar \to 0$ in the usual way. Now we know that $r = k + 2$ and that the level of the theory, k, plays the same role as $1/\hbar$. Consequently for $\hbar \to 0$ we have $k \to \infty$ and thus $r \to \infty$. This means Cos π/r goes to unity and we are left with $\left|V_L e^{2\pi i/r}\right| \leq (2)^{n-1}$. This means for a space behavior ($n = 3$) one finds $\left|V_L e^{2\pi i/r}\right| \leq d_c^{(3)} = 4$, which is our classical spacetime dimension indicating that one dimension ($4 - 3 = 1$) will remain invisible in the three-dimensional space giving rise to $n = 3 + 1$ spacetime. It is also clear that $2 \cos \pi/r = 2$ corresponds in the bijection formula to $d_c^{(0)} = 1/2$. In turn, for $d_c^{(0)} = 1/2$ we have $\sim\langle n\rangle = 3$ and $\langle d_c\rangle = 4$, which reinforces our conclusion of why classical spacetime is $3 + 1$ rather than simply 4 dimensional [21], [22].

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